Design of an Improved Computational Model for Traffic Flow

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Abstract

Transport emissions are one of the major contributors to the rise in air pollution today. In several large urban areas, air pollution from traffic congestion contributes to more than 2200 premature deaths annually and costs the health system at least $18 billion. The development of models that allow the effective prediction of congestion and emissions on critical traffic corridors is crucial in advancing and evaluating solutions that aim to mitigate environmental and health consequences of traffic. We adopt a numerical analysis approach and utilize partial differential equations (PDEs) as we are interested in exploring the predictive capability of non linear PDE based traffic models.

One lane model

One of the earlier traffic models is known as the Lighthill-Whitham-Richards (LWR) model. The LWR model was developed for unidirectional traffic on a single road ([3, 4]) and is given by:

$$\begin{aligned} u_t + (v(u)u)_x = 0 \\ u(x,0) = u_0(x) \text{ and } u(0,t) = \alpha \text{ for } \alpha \in \mathbb{R} \end{aligned} \tag{1}$$

- $u(x,t)$ denotes the density of cars at position x on the road at time $t \geq 0$.
- $u_0(x)$ is the initial condition that models the initial distribution of density of cars on the road.
- $u(0,t) = \alpha$ indicates what the density of cars is at the entrance of the road.
- $v(u)$ denotes the velocity of cars on the road as a function of density of cars u and expresses at what speed cars drive when the road has a density u of cars on the road. The LWR model uses a linear velocity function. We develop a non-linear velocity function, using a particle-based model, to better portray velocity-density relations:

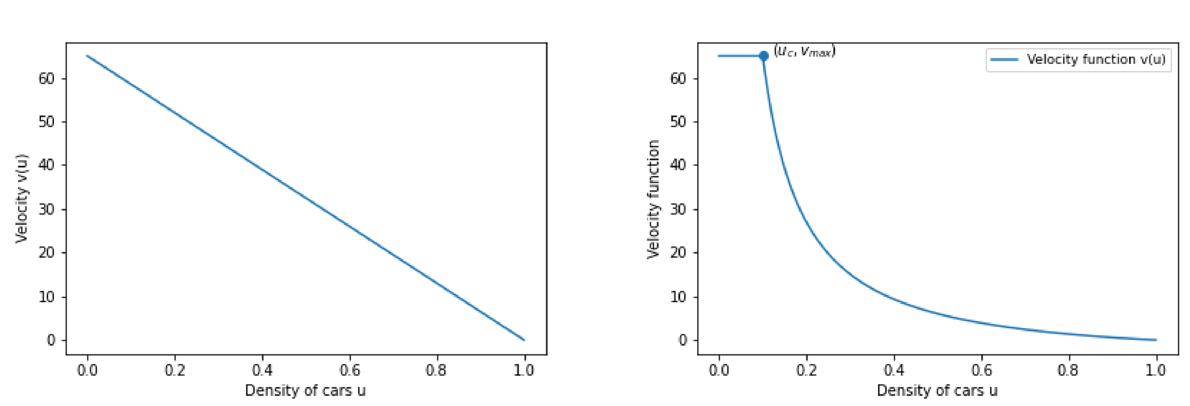

Figure 1. Plot of the linear velocity function $v(u) = v_{max}(1-u)$ and a non-linear velocity function satisfying realistic driver behaviors with $v_{max} = 65$

Conditions on velocity functions for stability

The velocity functions needs to be **monotonically non increasing** and **differentiable**.

We choose to ensure well-posedness of our problem through L^1 stability. Given two entropy solutions $u(x,t)$ and $\tilde{u}(x,t)$ of 1 with initial conditions $u(x,0)$ and $\tilde{u}(x,0)$ respectively, the solutions have to satisfy L_1 contractivity [2], that is:

$$\|u(x,t) - \tilde{u}(x,t)\|_{L_1} \leq \|u(x,0) - \tilde{u}(x,0)\|_{L_1} \tag{2}$$

Differentiability and monotonicity are sufficient conditions to ensure L_1 contractivity even for the multi-lane model including source terms [1].

Particle-based model and Modeling driver behaviors

We advance a particle-based model which uses Object Oriented Programming to model cars and roads to simulate various driving situations to generate a velocity-density function. To update a Car over time, there are various criteria that we consider:

1. The second rule of space between a car and its neighbor: In California, it is recommended to maintain a 3 second rule of space between a car and its neighbor.
2. The variation between the velocity of a Car and the maximum speed limit on the road v_{max}.
3. The variation between a Car's velocity and the Car's neighbor's velocity.
4. The variation between a Car's acceleration and the Car's neighbor's acceleration.

We incorporate various driver behaviors:

1. Abidance by drivers of the maximum speed.
2. Focus uniquely on the front car.
3. Maintaining a second rule of space between cars depending on their level of risk aversion.
4. Risk aversion in the intensity of acceleration.

Velocity-density profiles with various driver behaviors

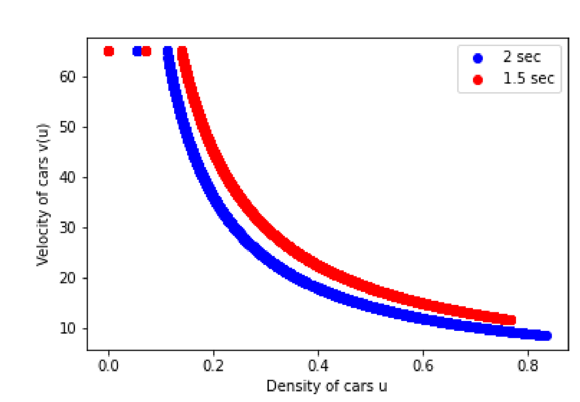

Figure 2. Comparison of velocity-density functions with drivers following different second rules

Fitting function to the velocity profiles

We fit a differentiable and monotonically non-increasing velocity function $v(u)$ to ensure L^1 contractivity: $v(u) = (\bar{v} \star \phi_\epsilon)(u)$ where $\phi_\epsilon(x) = \frac{1}{\epsilon}\phi(\frac{x}{\epsilon})$, $\phi(x)$ a standard symmetric Friedrichs mollifier and

$$\bar{v}(u) = \begin{cases} v_{max} & \text{if } u \leq u_c \\ K_1\left(\frac{1}{u^{K_2}-1}\right)^{K_3} & \text{if } x > u_c \end{cases} \tag{3}$$

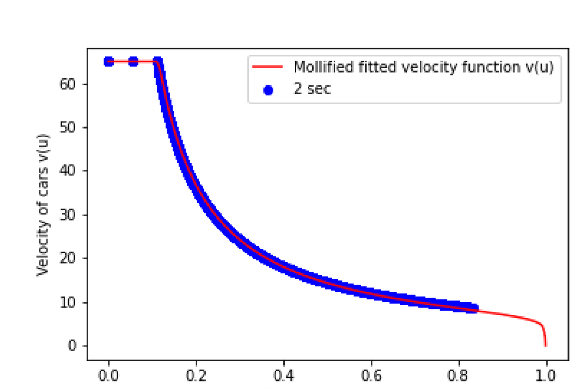

Figure 3. Velocity-density data points with $s_0 = 2.0$ along with the mollified fitted velocity function with $K_1 = 7$ and $K_2 = 7.0$, $\epsilon = 0.01$ and $u_c = 0.115$ and $K_3 = \frac{\log(\frac{v_{max}}{K_1})}{\log(\frac{1}{u_c^{K_2}} - 1)}$

PDE model results

Comparison between solutions of the PDEs with a linear and non-linear velocity:

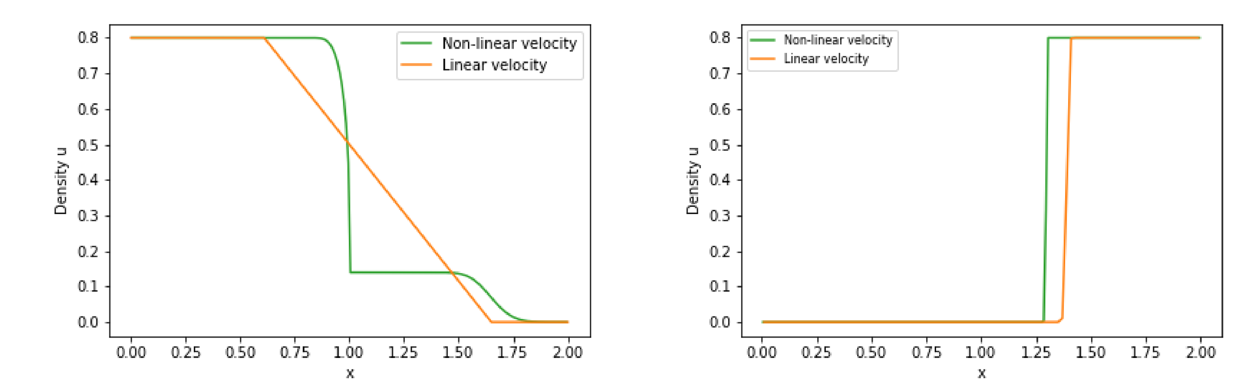

Figure 4. Solution of different Riemann problems where for the first problem, we have $u_L = 0.8$, $u_R = 0$ and $u_{bdy} = 0.8$ at $t = 0.01$ and for the second problem, we have $u_L = 0$, $u_R = 0.8$ and $u_{bdy} = 0$ at $t = 0.03$ using linear and non-linear velocity functions

Comparison between solutions of non-linear PDEs with different driver behaviors:

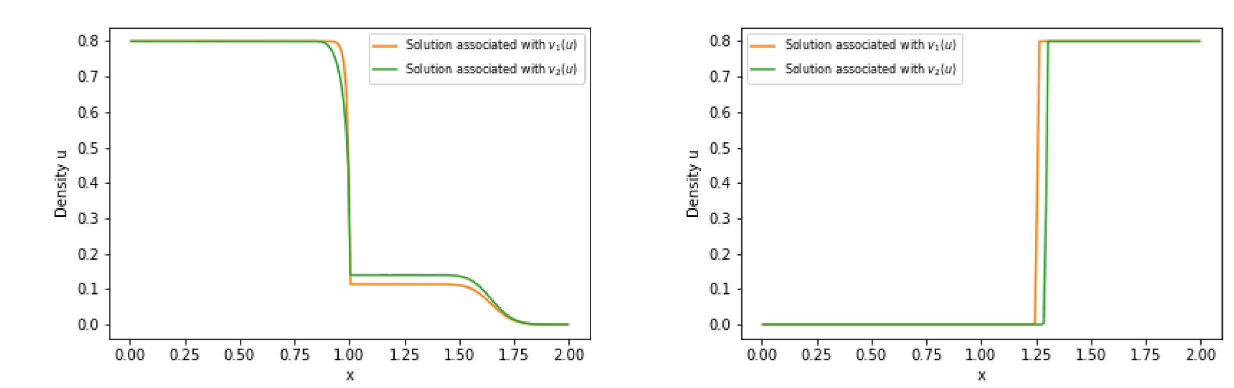

Figure 5. Solution of different Riemann problems where for the first problem, we have $u_L = 0.8$, $u_R = 0$ and $u_{bdy} = 0.8$ at $t = 0.01$ and for the second problem, we have $u_L = 0$, $u_R = 0.8$ and $u_{bdy} = 0$ at $t = 0.0e$ using non-linear velocity functions portraying different driver behaviors.

Discussion of the results

1. Using the particle-based model, we see that for drivers that are comfortable with a larger second rules, the point for which the velocity function start decreasing from v_{max} shifts to the left in Figure 2.
2. For the PDE solutions, using the linear and non-linear velocity functions, we obtain different solutions in 4:
   - For the first sub-plot, the rarefaction wave in the linear case assumes that all drivers in the traffic jam drive at the same speed which is not realistic. In the non-linear case, the expansion wave portrays how cars at the end of a traffic jam start driving fast first and the other cars follow at a later time.
   - For the second sub-plot, the shock in the linear case moves faster than the shock in the non-linear case.
3. Using the PDE model, we observe that we obtain distinct solutions in Figure 5 with velocity functions portraying different behaviors: different expansion and shock speeds of the solutions that illustrate various level of driving comforts, respectively.

References

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