Sparsified QR (spaQR) : A fast, sparse, approximate QR factorization

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Overview

- Fast, sparse QR factorization to solve
  - Sparse linear systems: $Ax = b, \ A \in \mathcal{R}^{N \times N}$
  - Sparse linear LS problems:
  - $\min_x \|Ax - b\|^2, A \in \mathcal{R}^{M \times N}, M \geq N$
- Based on a multifrontal QR factorization
- Matrix reordered using Nested Dissection

ND separators   Interfaces   Elimination tree

- Key assumption: Fill-in blocks are low rank
- Sparsification of one interface p:

$$A_p = \begin{bmatrix} A_{pp} & A_{pn} \\ A_{np} & A_{nn} \end{bmatrix} \rightarrow \begin{bmatrix} I & \bar{A}_{p_1 n} \\ & \bar{A}_{p_2 n} \\ \bar{A}_{np} & A_{nn} \end{bmatrix} \rightarrow \begin{bmatrix} I_f & & W_{fn}^{(2)} \\ & I_c & W_{cn}^{(2)} \\ & & W'_{cn} \\ & & W'_{fn} \\ W_{fn}^{(1)T} & W_{cn}^{(1)T} & A_{nn} \end{bmatrix}$$

using a rank revealing QR on,

$$A_{p_2 n} = \begin{bmatrix} Q'_{pc} & Q'_{pf} \end{bmatrix} \begin{bmatrix} W'_{cn} \\ W'_{fn} \end{bmatrix}$$

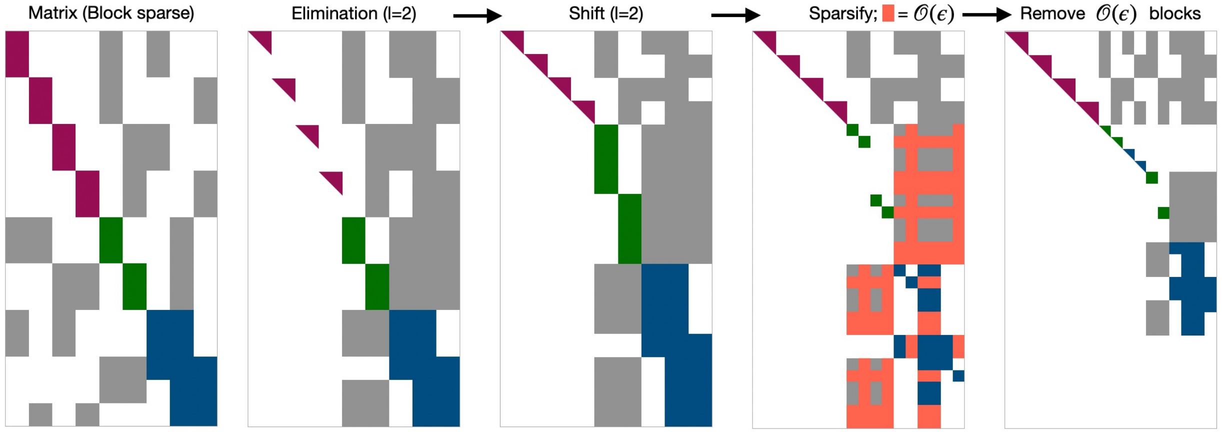
$$\begin{bmatrix} A_{np}^T & A_{p_1 n} \end{bmatrix} = \begin{bmatrix} Q_{pf} & Q_{pc} \end{bmatrix} \begin{bmatrix} W_{fn}^{(1)} & W_{fn}^{(2)} \\ W_{cn}^{(1)} & W_{cn}^{(2)} \end{bmatrix}$$

where, $\|W_{fn}^{(1)}\|_2 \approx \|W_{fn}^{(2)}\|_2 \approx \|W'_{fn}\|_2 = \mathcal{O}(\epsilon)$

Algorithm: Sparsified QR (spaQR)

Require: Matrix $A \in \mathbb{R}^{M \times N}$, $M \geq N$
Require: Tolerance ϵ
for all $l = L-1, L-2, \ldots 0$ **do**
1. Factorize separators at level l
2. Shift rows below diagonal of eliminated separators
3. Scale, Sparsify interfaces at all levels $l' > l$
4. Ignore $\mathcal{O}(\epsilon)$ blocks

end for



Results

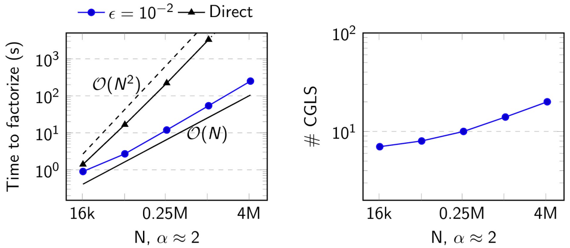
PDE constrained optimization: Solve $J^T x = b$
J is the Jacobian of size $\alpha M \times N$

$$\min_{u,z} \ \frac{1}{2}\int_\Omega (u - u_d)^2 dx + \frac{1}{2}\lambda \int_\Omega z^2 dx$$

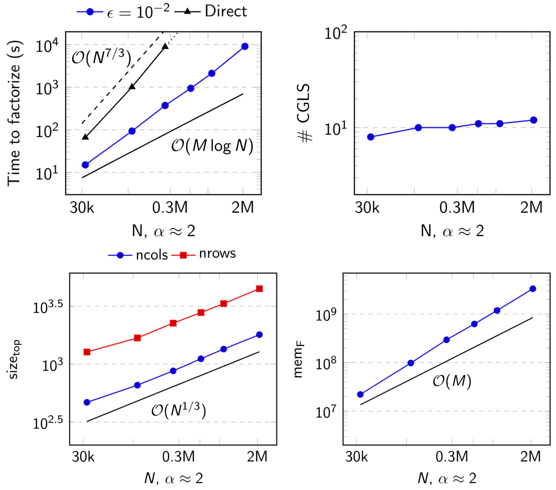
subject to $-\nabla \cdot (z \nabla u) = h$ in Ω

$u = 0$ in $d\Omega$

2D $n \times n$ grid, $N = n^2$



3D $n \times n \times n$ grid, $N = n^3$



Conclusions

- Produces an approx. factorization with tunable cost and accuracy
- Complexity: $\mathcal{O}(M \log N)$ in time and $\mathcal{O}(M)$ in space
- Exhibits more parallelism than direct methods
- Applications in CFD, PDE constrained optimization, computer graphics and vision.

References:
- A. Gnanasekaran and E. Darve, Hierarchical orthogonal factorization: Sparse Square Matrices, 2020
- A. Gnanasekaran and E. Darve, Hierarchical orthogonal factorization: Sparse Least Squares Problems, 202
- Code: https://github.com/Abeynaya/spaQR_public